Opérateurs. Formulaire.

Soit M un point de l'espace. On désigne par $U(M)$ une fonction scalaire et $\vec{A}(M)$ une fonction vectorielle.

- **Coordonnées cartésiennes.**

$$\overrightarrow{\text{grad}}\,U = \frac{\partial U}{\partial x}\vec{u}_x + \frac{\partial U}{\partial y}\vec{u}_y + \frac{\partial U}{\partial z}\vec{u}_z$$

$$\text{div}\,\vec{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

$$\overrightarrow{\text{rot}}\,\vec{A} = \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right)\vec{u}_x + \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right)\vec{u}_y + \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)\vec{u}_z$$

$$\Delta U = \frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} + \frac{\partial^2 U}{\partial z^2}$$

$$\overrightarrow{\Delta}\vec{A} = \left(\frac{\partial^2 A_x}{\partial x^2} + \frac{\partial^2 A_x}{\partial y^2} + \frac{\partial^2 A_x}{\partial z^2}\right)\vec{u}_x + \left(\frac{\partial^2 A_y}{\partial x^2} + \frac{\partial^2 A_y}{\partial y^2} + \frac{\partial^2 A_y}{\partial z^2}\right)\vec{u}_y + \left(\frac{\partial^2 A_z}{\partial x^2} + \frac{\partial^2 A_z}{\partial y^2} + \frac{\partial^2 A_z}{\partial z^2}\right)\vec{u}_z$$

- **Coordonnées cylindriques.**

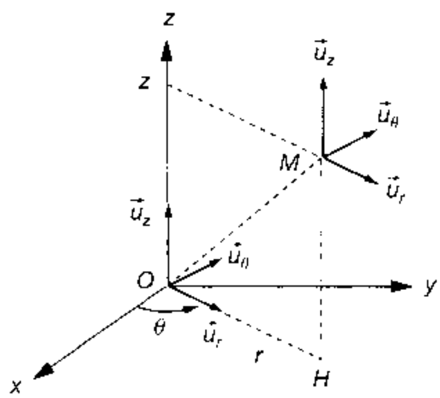
$$\overrightarrow{\text{grad}}\,U = \frac{\partial U}{\partial r}\vec{u}_r + \frac{1}{r}\frac{\partial U}{\partial \theta}\vec{u}_\theta + \frac{\partial U}{\partial z}\vec{u}_z$$

$$\text{div}\,\vec{A} = \frac{1}{r}\frac{\partial (rA_r)}{\partial r} + \frac{1}{r}\frac{\partial A_\theta}{\partial \theta} + \frac{\partial A_z}{\partial z}$$

$$\overrightarrow{\text{rot}}\,\vec{A} = \left(\frac{1}{r}\frac{\partial A_z}{\partial \theta} - \frac{\partial A_\theta}{\partial z}\right)\vec{u}_r + \left(\frac{\partial A_r}{\partial z} - \frac{\partial A_z}{\partial r}\right)\vec{u}_\theta + \left(\frac{1}{r}\frac{\partial (rA_\theta)}{\partial r} - \frac{1}{r}\frac{\partial A_r}{\partial \theta}\right)\vec{u}_z$$

$$\Delta U = \frac{\partial^2 U}{\partial r^2} + \frac{1}{r}\frac{\partial U}{\partial r} + \frac{1}{r^2}\frac{\partial^2 U}{\partial \theta^2} + \frac{\partial^2 U}{\partial z^2}$$

Remarque : $\frac{\partial^2 U}{\partial r^2} + \frac{1}{r}\frac{\partial U}{\partial r} = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial U}{\partial r}\right)$



- **Coordonnées sphériques.**

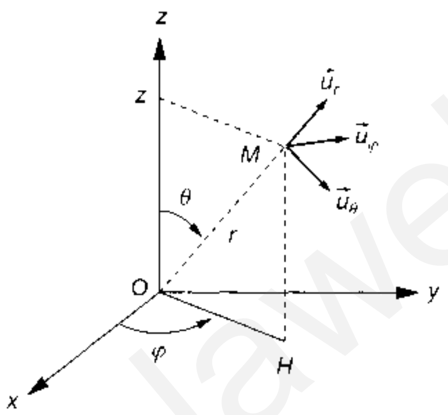
$$\overrightarrow{\mathrm{grad}}U = \frac{\partial U}{\partial r}\vec{u}_r + \frac{1}{r}\frac{\partial U}{\partial \theta}\vec{u}_\theta + \frac{1}{r\sin\theta}\frac{\partial U}{\partial \varphi}\vec{u}_\varphi$$

$$\mathrm{div}\,\vec{A} = \frac{1}{r^2}\frac{\partial\left(r^2 A_r\right)}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial\left(\sin\theta A_\theta\right)}{\partial\theta} + \frac{1}{r\sin\theta}\frac{\partial A_\varphi}{\partial\varphi}$$

$$\overrightarrow{\mathrm{rot}}\,\vec{A} = \left\{\frac{1}{r\sin\theta}\left(\frac{\partial\left(\sin\theta A_\varphi\right)A_z}{\partial\theta} - \frac{\partial A_\theta}{\partial\varphi}\right)\right\}\vec{u}_r + \left\{\frac{1}{r}\left(\frac{1}{\sin\theta}\frac{\partial A_r}{\partial\varphi} - \frac{\partial\left(rA\varphi\right)}{\partial r}\right)\right\}\vec{u}_\theta + \left\{\frac{1}{r}\left(\frac{\partial\left(rA_\theta\right)}{\partial r} - \frac{\partial A_r}{\partial\theta}\right)\right\}\vec{u}_\varphi$$

$$\Delta U = \frac{\partial^2 U}{\partial r^2} + \frac{2}{r}\frac{\partial U}{\partial r} + \frac{1}{r^2\sin\theta}\frac{\partial\left(\sin\theta\frac{\partial U}{\partial\theta}\right)}{\partial\theta} + \frac{1}{r^2\sin^2\theta}\frac{\partial\, U}{\partial\varphi^2}$$

Remarque : $\frac{\partial^2 U}{\partial r^2} + \frac{2}{r}\frac{\partial U}{\partial r} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial U}{\partial r}\right)$



- **Calcul vectoriel intégral.**

On considère un contour C fermé et orienté, sur lequel s'appuie une surface Σ ouverte. Soient $\mathrm{d}\vec{l}$ un déplacement élémentaire sur C et $\mathrm{d}\vec{S}$ un élément de surface de Σ orienté par le sens de C.

$$\oint_C \vec{A}.\mathrm{d}\vec{l} = \iint_S \overrightarrow{\mathrm{rot}}\vec{A}.\mathrm{d}\vec{S} \qquad \textit{Théorème de STOKES} - \textit{AMPERE}$$

$$\oint_C U\mathrm{d}\vec{l} = -\iint \overrightarrow{\mathrm{grad}}U \wedge \mathrm{d}\vec{S}$$

On considère un volume V délimité par une surface fermée Σ .Soit $\mathrm{d}\vec{S}$ un élément de surface de Σ orienté vers l'extérieur de V.

$$\oiint_\Sigma \vec{A}.\mathrm{d}\vec{S} = \iiint_V \mathrm{div}\vec{A}\,\mathrm{d}V \qquad \textit{Théorème de GREEN} - \textit{OSTROGRADSKY}$$

$$\oiint_\Sigma U\,\mathrm{d}\vec{S} = \iiint_V \overrightarrow{\mathrm{grad}}U\,\mathrm{d}V$$

$$\oiint_\Sigma \vec{A}\wedge\mathrm{d}\vec{S} = -\iiint_V \overrightarrow{\mathrm{rot}}\vec{A}\,\mathrm{d}V$$

- **Opérations.**

$$\overrightarrow{\text{grad}}(UV) = U\,\overrightarrow{\text{grad}}V + V\,\overrightarrow{\text{grad}}U$$

$$\text{div}\left(U\vec{A}\right) = U\text{div}\vec{A} + \left(\overrightarrow{\text{grad}}U\right).\vec{A}$$

$$\text{div}\left(\vec{A}\wedge\vec{B}\right) = \vec{B}.\overrightarrow{\text{rot}}\vec{A} - \vec{A}.\overrightarrow{\text{rot}}\vec{B}$$

$$\overrightarrow{\text{rot}}\left(U\vec{A}\right) = U\,\overrightarrow{\text{rot}}\vec{A} + \left(\overrightarrow{\text{grad}}U\right)\wedge\vec{A}$$

$$\overrightarrow{\text{rot}}\left(\vec{A}\wedge\vec{B}\right) = \vec{A}\text{div}\vec{B} - \vec{B}\text{div}\vec{A} + \left(\vec{B}.\overrightarrow{\text{grad}}\right)\vec{A} - \left(\vec{A}.\overrightarrow{\text{grad}}\right)\vec{B}$$

$$\overrightarrow{\text{rot}}\,\overrightarrow{\text{grad}}U = \vec{0}$$

$$\text{div}\,\overrightarrow{\text{rot}}\vec{A} = 0$$

$$\text{div}\overrightarrow{\text{grad}}U = \Delta U$$

$$\overrightarrow{\text{rot}}\,\overrightarrow{\text{rot}}\vec{A} = \overrightarrow{\text{grad}}\,\text{div}\vec{A} - \vec{\Delta}\vec{A}$$

$$\text{d}U = \overrightarrow{\text{grad}}U.\text{d}\vec{l}$$